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Selfduality of $d = 2$ Reduction of Gravity Coupled to a σ-Model

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Abstract

Dimensional reduction in two dimensions of gravity in higher dimension, or more generally of $d = 3$ gravity coupled to a σ-model on a symmetric space, is known to possess an infinite number of symmetries. We show that such a bidimensional model can be embedded in a covariant way into a σ-model on an infinite symmetric space, built on the semidirect product of an affine group by the Witt group. The finite theory is the solution of a covariant selfduality constraint on the infinite model. It has therefore the symmetries of the infinite symmetric space. (We give explicit transformations of the gauge algebra.) The usual physical fields are recovered in a triangular gauge, in which the equations take the form of the usual linear systems which exhibit the integrable structure of the models. Moreover, we derive the constraint equation for the conformal factor, which is associated to the central term of the affine group involved.

1 Introduction

Dimensional reduction of gravity in dimension $d = 4$ to $d = 2$ enlarges the group of symmetries to an infinite group [1], which has been related to the integrable structure of the theory [2, 3]. The symmetry group appeared to be an affine Kac-Moody group [4, 5, 6]. Furthermore, it was shown that this structure was shared by a full class of theories, such as supergravities, which in dimension three reduce (for the bosonic sector) to a symmetric space σ-model coupled to gravity [4, 5, 7].

In addition to the scalars which describe a $\mathfrak{G}/\mathfrak{H}$ σ-model, the (bosonic) degrees of freedom of these models consist of a dilaton ρ and the conformal factor $\lambda = e^{\sigma}$ of the metric. The equations of motion are, in conformal gauge,

$$\mathrm{d}{*}\mathrm{d}\rho = 0 \qquad (1.1)$$

$$\nabla(\rho * P) = 0 \qquad (1.2)$$

(P and ∇ are precisely defined below.) with in addition a first order constraint on the conformal factor,

$$\partial_\pm\rho\,\partial_\pm\hat{\sigma} = \frac{1}{2}\rho\,\langle P_\pm, P_\pm\rangle \qquad (1.3)$$

where $\hat{\sigma} = \sigma - \frac{1}{2}\ln(\partial_+\rho\,\partial_-\rho)$. We choose a Lorentzian spacetime for exposition; it is not hard to adapt the results for a Euclidean one.

In [8, 9], the symmetry of such a theory was enlarged to the full semidirect product of the Witt group and an affine Kac-Moody group. The fields are infinitely dualised and live in an infinite coset

$$\mathcal{M} = \frac{\mathcal{W} \ltimes \mathfrak{G}^\infty}{\mathcal{K} \ltimes \mathfrak{H}^\infty}\ . \tag{1.4}$$

In this formalism, the equations of motion come from linear systems which are imposed as constraints in a triangular gauge.

Here, following [10], which deals with the flat space σ-model, we restore the infinite $\mathcal{K} \ltimes \mathfrak{H}^\infty$ gauge-invariance: we define the finite, constrained model through a covariant selfduality equation on the infinite tower of fields. Previously known Lax pairs are recovered as consequences of this constraint when we go into the triangular gauge. This gives a formulation of the $d = 2$ theory very analogous to the oxidised versions ($d \geq 3$) [11, 12, 13].

We also derive the constraint for the conformal factor from the selfduality constraint. Whereas for other fields the duality involves a Hodge dualisation, it is worth noticing that this is not the case for the conformal factor, associated to the central term of the group.

In section 2, we decribe the infinite symmetric space $\mathcal{M}$ and the algebraic structures involved. In section 3, we define the duality operator and we show that a a selfduality constraint can be imposed and is covariant with respect to the infinite gauge algebra transformations. Finally, we fix the gauge in section 4 to recover the physical content of the theory, and we derive the Lax pair equations associated to the dynamical fields of the model, together with the conformal factor constraint.

2 Infinite σ-model structure

Following [8], we consider fields in an infinite-dimensional symmetric space

$$\mathcal{M} = \frac{\mathcal{W} \ltimes \mathfrak{G}^\infty}{\mathcal{K} \ltimes \mathfrak{H}^\infty}\ . \tag{2.1}$$

$\mathcal{W}$ is the "group" of diffeomorphisms of the real line, $\mathfrak{G}^\infty$ is the affine extension $\mathfrak{G}^{(1)}$ of the simple group $\mathfrak{G}$ and $\mathcal{K} \ltimes \mathfrak{H}^\infty$ is the subgroup of fixed points of $\mathcal{W} \ltimes \mathfrak{G}^\infty$ under some involution.

Explicitely, $\mathfrak{G}^\infty$ is the set of pairs $(g(t), a)$ where $g(t)$ is a map from $\mathbb{R}_+^\times$ to $\mathfrak{G}$ and a is a positive real number. The group law is

$$(g_1(t), a_1)(g_2(t), a_2) = \left(g_1(t)g_2(t), a_1 a_2 e^{\Omega(g_1,g_2)}\right) \tag{2.2}$$

where Ω is a group 2-cocycle (see [6]). The Lie algebra is the affine Kac-Moody algebra $\mathfrak{g}^\infty$ with analytic functions $b(t)$ with values in $\mathfrak{g}$ and in addition a central charge c; commutation relations are

$$[b_1(t), b_2(t)] = [b_1(t), b_2(t)]_{\mathfrak{g}} + \omega(b_1, b_2)\, c \tag{2.3}$$

where ω is a 2-cocycle of the loop algebra (see [6, 14]). Here, it is defined as

$$\omega(b_1, b_2) = \frac{1}{2}\oint_{\mathcal{C}_1} \mathrm{d}t\, \langle \partial_t b_1(t), b_2(t)\rangle + \frac{1}{2}\oint_{\mathcal{C}_2} \mathrm{d}t\, \langle \partial_t b_1(t), b_2(t)\rangle \tag{2.4}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are two contours exchanged and reversed under $t \to \frac{1}{t}$, avoiding singularities [6].

$\mathcal{W}$ is, at least formally, the group $\mathrm{Diff}^+(\mathbb{R}_+^\times)$ of analytic diffeomorphisms of the real line preserving the orientation. We see it as Laurent series $f(t)$ with the law group given by composition:

$$(f_1 \circ f_2)(t) = f_1(f_2(t)) \ . \tag{2.5}$$

Its Lie algebra is the real Witt algebra with generators $L_n = t^{n+1}\partial_t$ $(n \in \mathbb{Z})$ and commutation relations

$$[L_m, L_n] = (n-m)L_{m+n} \ . \tag{2.6}$$

The semidirect product $\mathcal{W} \ltimes \mathfrak{G}^\infty$ is given by triples $(f, g, a) \in \mathcal{W} \times \mathfrak{G}^\infty$ with product law

$$(f_1, g_1, a_1)(f_2, g_2, a_2) = \Big(f_1 \circ f_2 \, , \, (g_1 \circ f_2)\, g_2 \, , \, a_1 a_2 e^{\Omega(g_1 \circ f_2, g_2)}\Big) \ . \tag{2.7}$$

The subgroup $\mathcal{K} \ltimes \mathfrak{H}^\infty$ consists of fixed points under an involution $\tau_\ltimes$, which is given by an involution on $\mathcal{W}$

$$\tau_{\mathcal{W}} : \ f(t) \ \longrightarrow \ \frac{1}{f(1/t)} \tag{2.8}$$

compatible with an involution on $\mathfrak{G}^\infty$

$$\tau_\infty : \ (g(t), a) \ \longrightarrow \ \left(\tau\left(g\left(\frac{1}{t}\right)\right), \frac{1}{a}\right) \ , \tag{2.9}$$

where τ is the involution fixing $\mathfrak{H}$. Denoting by $\mathcal{K}$ and $\mathfrak{H}^\infty$ the sets of fixed points under respectively $\tau_{\mathcal{W}}$ and τ_∞, the fixed points of $\mathcal{W} \ltimes \mathfrak{G}^\infty$ under $\tau_\ltimes$ is $\mathcal{K} \ltimes \mathfrak{H}^\infty$. It consists of elements $(f, g, 1)$ with $f\left(\frac{1}{t}\right) = \frac{1}{f(t)}$ and $g\left(\frac{1}{t}\right) = \tau(g(t))$.

On the Lie algebra side, this gives an involution acting on generators as

$$\tau_\ltimes : \left\{ \begin{array}{ccc} L_n & \longrightarrow & -L_{-n} \\ t^n T & \longrightarrow & t^{-n}\tau(T) \\ c & \longrightarrow & -c \end{array} \right. \ . \tag{2.10}$$

From a field in two dimensions with values in the infinite dimensional coset,

$$\mathcal{V}(x) \in \frac{\mathcal{W} \ltimes \mathfrak{G}^\infty}{\mathcal{K} \ltimes \mathfrak{H}^\infty} \ , \tag{2.11}$$

we derive the pull-back of the Maurer-Cartan form

$$\mathcal{G} = \mathrm{d}\mathcal{V}\,\mathcal{V}^{-1} \tag{2.12}$$

with Ω' the mixed cocycle defined in [6]. $\mathcal{G}$ satisfies the Maurer-Cartan equation

$$\mathrm{d}\mathcal{G} = \mathcal{G} \wedge \mathcal{G} \ . \tag{2.13}$$

3 Selfduality

We decompose $\mathcal{G}$ as

$$\mathcal{G} = \mathcal{X} + \mathcal{Y} \tag{3.1}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are respectively invariant and anti-invariant under $\tau_\ltimes$. Under a gauge transformation, $\mathcal{X}$ behaves as a gauge field, whereas $\mathcal{Y}$ is covariant (see [10] for details). Note that $\mathcal{X}$ and $\mathcal{Y}$ are formal series in t and t^{-1} and do not necessarily make sense as analytic functions of t.

In order to impose a selfduality constraint, we define an involution $\mathcal{S}$ acting as

$$\mathcal{S}: \left\{ \begin{array}{ccc} \alpha\, L_n & \longrightarrow & -*\alpha\, L_{1-n} \\ \beta\, t^n T & \longrightarrow & *\beta\, t^{1-n}\tau(T) \\ \gamma\, c & \longrightarrow & -\gamma\, c \end{array} \right. \tag{3.2}$$

where α, β, γ are 1-forms and T is any generator of $\mathfrak{g}$.

Our claim is the following: the selfduality constraint

$$\mathcal{S}\mathcal{Y} = \mathcal{Y} \tag{3.3}$$

reduces the infinite-dimensional σ-model to the $d = 2$ model obtained by dimensional reduction of a $d = 3$ $\mathfrak{G}/\mathfrak{H}$ σ-model coupled to gravity. Moreover, this constraint is invariant under a global action of $\mathcal{W} \ltimes \mathfrak{G}^\infty$ on the right and a local $\mathcal{K} \ltimes \mathfrak{H}^\infty$ gauge transformation on the left.

The global invariance is trivial: $\mathcal{G} = \mathrm{d}\mathcal{V}\,\mathcal{V}^{-1}$ is invariant under right multiplication of $\mathcal{V}$ by any constant group element Λ

$$\mathcal{V}(x) \;\longrightarrow\; \mathcal{V}(x)\Lambda\ . \tag{3.4}$$

Let us check the gauge invariance, with respect to left multiplication by elements of $\mathcal{K} \ltimes \mathfrak{H}^\infty$:

$$\mathcal{V}(x) \;\longrightarrow\; H(x)\mathcal{V}(x)\ . \tag{3.5}$$

If we write $\mathcal{G}$ as

$$\mathcal{G} = \sum_{n\in\mathbb{Z}} A_n L_n + \sum_{n\in\mathbb{Z}} B_n t^n + C\, c \tag{3.6}$$

with $A_n \in \mathbb{R}$, $B_n \in \mathfrak{g}$ and $C \in \mathbb{R}$, we have explicitely

$$\mathcal{Y} = \frac{1}{2}\sum_{n\in\mathbb{Z}}(A_n + A_{-n})L_n + \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(B_n - \tau(B_{-n})\right) t^n + C\, c\ . \tag{3.7}$$

Solutions of (3.3) are given by

$$\begin{array}{rcl} A_n + A_{-n} & = & 2 *^n A_0 \\ B_n - \tau(B_{-n}) & = & 2 *^n P \\ C & = & 0 \end{array} \tag{3.8}$$

where we decompose $B_0 = P + Q$ with P and Q respectively anti-invariant and invariant under the involution τ, and with $*^2 = 1$ in Lorentzian signature.

We consider first an infinitesimal gauge transformation in Lie($\mathcal{K}$): $\delta k = \delta a(x)(L_p - L_{-p})$. It acts on fields as

$$\begin{array}{rcl} A_n & \longrightarrow & A_n + (n-2p)\,\delta a\, A_{n-p} - (n+2p)\,\delta a\, A_{n+p} + (\delta_{n,p} - \delta_{n,-p})\,\mathrm{d}\delta a \\ B_n & \longrightarrow & B_n + (n-p)\,\delta a\, B_{n-p} - (n+p)\,\delta a\, B_{n+p} \\ C & \longrightarrow & C \end{array} \tag{3.9}$$

This leave (3.8) invariant, with

$$\begin{array}{rcl} A_0 & \longrightarrow & A_0 - 4p\,\delta a *^p A_0 \\ P & \longrightarrow & P - 2p\,\delta a *^p P\ . \end{array} \tag{3.10}$$

For an infinitesimal transformation $\delta h = \delta b(x) t^p + \tau(\delta b(x)) t^{-p} \in \mathfrak{h}^\infty$, the fields transforms as

$$\begin{array}{rcl} A_n & \longrightarrow & A_n \\ B_n & \longrightarrow & B_n + [\delta b, B_{n-p}] + [\tau(\delta b), B_{n+p}] + (\delta_{n,p} - \delta_{n,-p})\,\mathrm{d}\delta b \\ C & \longrightarrow & C + \omega\,(\delta h, B) \end{array} \tag{3.11}$$

with $B = \sum_{n\in\mathbb{Z}} B_n t^n$.

It is not hard to check that the A_n and B_n parts of (3.8) are invariant under these transformations, with

$$\begin{array}{rcl} A_0 & \longrightarrow & A_0 \\ P & \longrightarrow & P + [\delta b + \tau(\delta b) \,, *^p P] \ . \end{array} \tag{3.12}$$

For the central extension, it is more subtle. Using (3.8), we can write down two expansions of B:

$$B \;=\; Q + P + \sum_{n\geq 1} t^n *^n P + \sum_{n\geq 1} \left(B_{-n} t^{-n} + \tau(B_{-n}) t^n\right) \tag{3.13}$$

$$B \;=\; Q + P + \sum_{n\geq 1} t^{-n} *^n P + \sum_{n\geq 1} \left(B_n t^n + \tau(B_n) t^{-n}\right) \tag{3.14}$$

In order to have some analytic quantity, we require that the last term of either (3.13) or (3.14) can be summed up for some value of $|t| \neq 1$ (it depends on the gauge). From the structure of this term, it will then be analytic in a annulus $\theta < |t| < \frac{1}{\theta}$. Let us suppose this is the case for (3.13). (The reasonning is analogous in the other case.) For $\theta < |t| < 1$, the full expansion makes sense as an analytic function, and we have

$$B = Q + \frac{1+t^2}{1-t^2} P + \frac{2t}{1-t^2} *P + \sum_{n\geq 1} \left(B_{-n} t^{-n} + \tau(B_{-n}) t^n\right) \ . \tag{3.15}$$

This is the analytic function which is used to compute the central extension in (3.11).

As we have seen, the contour integral used for computing $\omega(\,\cdot\,,\,\cdot\,)$ is in fact the average over two contours (with the same orientation) exchanged and reversed under $t \to \frac{1}{t}$. The last term of (3.15) is regular in $|t| = 1$, so we can take the single unit circle. For the terms singular in $t = \pm 1$, we choose a pair of contours avoiding the singular points; practically, we average on the residue at 0 and ∞. We find

$$\omega(\delta h, B) = 0 \tag{3.16}$$

and therefore (3.8) is completely invariant under δh.

The selfduality constraint is thus invariant under the full gauge algebra.

4 Physical content

In order to recover the physical content of the selfdual theory, we fix (partially) the gauge for $\mathcal{V}$ in the following way:

$$\mathcal{V} = (\mathrm{id}, 1, a)(\mathrm{id}, g, 1)(f, 1, 1) = (f, g\circ f, a) \tag{4.1}$$

with f and g regular in $t = 0$, and $f(0) = 0$ (see [8]). This triangular gauge will allow to recover physical fields of the constrained model as $t = 0$ values of fields.

From (2.12), we get

$$\mathcal{G} = \left(\mathrm{d}f\circ f^{-1} \,,\, \mathrm{d}g\, g^{-1} + \partial_t g\, g^{-1} \mathrm{d}f\circ f^{-1} \,,\, \mathrm{d}a\, a^{-1} - \Omega'(g^{-1}, \mathrm{d}g\, g^{-1} + \partial_t g\, g^{-1} \mathrm{d}f\circ f^{-1})\right) \ . \tag{4.2}$$

As f and g are regular in 0, we have $A_n = 0$ and $B_n = 0$ in (3.6) for $n < 0$.

Defining

$$\begin{array}{rcl} \rho & = & \partial_t f\,|_{t=0} \\ g_0 & = & g\,|_{t=0} \\ \hat{\sigma} & = & \ln(a) \end{array} \tag{4.3}$$

we have from (4.2)

$$
\begin{aligned}
A_0 &= \mathrm{d}\rho\,\rho^{-1} \\
B_0 &= \mathrm{d}g_0\,g_0^{-1} \\
C &= \mathrm{d}a\,a^{-1} - \Omega'(g^{-1}, B)\ .
\end{aligned}
\tag{4.4}
$$

Plugging the solutions (3.8) of the selfduality constraint $\mathcal{S}\mathcal{Y} = \mathcal{Y}$ into (4.2), we get

$$
\mathrm{d}f\circ f^{-1} = \frac{1+t^2}{1-t^2}\,\mathrm{d}\rho\,\rho^{-1}\,t\partial_t + \frac{2t}{1-t^2}\,{*\mathrm{d}\rho}\,\rho^{-1}\,t\partial_t \tag{4.5}
$$

$$
(\mathrm{d}(g\circ f)\circ f^{-1})g^{-1} = Q + \frac{1+t^2}{1-t^2}\,P + \frac{2t}{1-t^2}\,{*P} \tag{4.6}
$$

$$
\mathrm{d}\hat{\sigma} = \Omega'\left(g^{-1}\,,\,Q + \frac{1+t^2}{1-t^2}\,P + \frac{2t}{1-t^2}\,{*P}\right)\ . \tag{4.7}
$$

(4.5) and (4.6) are the linear systems associated to the $d=2$ reduction of the $d=3$ $\mathfrak{G}/\mathfrak{H}$ σ-model coupled to gravity [5, 6, 8]. ρ is the dilaton, and $\sigma = \hat{\sigma} + \frac{1}{2}\ln(\partial_+\rho\,\partial_-\rho)$ is the Liouville variable. $t = f(s)$ is often called the "variable spectral parameter".

The equations of motion of ρ and g_0 come from the Maurer-Cartan equation $\mathrm{d}\mathcal{G} = \mathcal{G}\wedge\mathcal{G}$:

$$
\mathrm{d}{*\mathrm{d}\rho} = 0 \tag{4.8}
$$

$$
\nabla(\rho\,{*P}) = 0 \tag{4.9}
$$

with the $\mathfrak{H}$-covariant derivative $\nabla = \mathrm{d} + [Q,\ \cdot\ ]$.

Finally, let us show that (4.7) is the constraint equation for the conformal factor. First, we turn to conformal coordinates $x^{\pm}$. Acting on the right with some constant element, we can always manage to have g infinitesimal in the vicinity of some spacetime point x_0: $g = 1 + \delta g$. In this region, (4.7) can be written as

$$
\partial_\pm\hat{\sigma} = \frac{1}{2}\,\omega\left(-\delta g\,,\,\frac{1\mp t}{1\pm t}\,P_\pm\right)\ . \tag{4.10}
$$

According to the definition of ω, we must take the average of the residues of $-\partial_t\delta g\,\frac{1\mp t}{1\pm t}\,P_\pm$ at 0 and ∞. In this case, it means we must take the residue at 0 plus one half of the residue at ±1 (δg is a regular function of t [6]). We get

$$
\partial_\pm\hat{\sigma} = \frac{1}{2}\,\langle \mp\,\partial_t\delta g\,|_{t=\mp1}\,,\,P_\pm\rangle\ . \tag{4.11}
$$

From the regularity of g in t, we also know that the poles in (4.6) comes from $\partial_t g\,g^{-1}\mathrm{d}f\circ f^{-1}$ on the left handside, here with $g = 1+\delta g$. Using also (4.5), we have

$$
\mp\,\partial_t\delta g\,|_{t=\mp1}\ \partial_\pm\rho\,\rho^{-1} = P_\pm\ . \tag{4.12}
$$

Combining this with (4.11), we recover the constraint for the conformal factor:

$$
\partial_\pm\rho\,\partial_\pm\hat{\sigma} = \frac{1}{2}\rho\,\langle P_\pm, P_\pm\rangle\ . \tag{4.13}
$$

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